Indian Statistical Institute, Bangalore Centre
B.Math. (I Year): 2008-2009
Semester II: Semestral Examination
Probability Theory - II

8.5.2009 Time : 3 hrs. Maximum Marks : 100

Note : This paper carries 108 marks. Any score above 100 will be treated as 100.

1. [6+10+8 marks] Let (X, Y) be a two dimensional absolutely continuous random variable with probability density function

$$\begin{aligned} f(x,y) &= \lambda^2 \exp(-\lambda y), \quad \text{if } \ 0 < x < y < \infty, \\ &= 0, \quad \text{otherwise}, \end{aligned}$$

where $\lambda > 0$ is a constant.

(i) Find the marginal densities.

(ii) Let $W = X, Z = Y - X$. Show that W, Z are independent random variables having the same distribution.

(iii) Let V_1, V_2 denote the order statistics corresponding to W, Z. Find the joint probability density function of V_1, V_2.

2. [10 marks] Let (X, Y) be a two dimensional absolutely continuous random variable with probability density function f. Show that the probability density function h of $(X + Y)$ is given by

$$h(z) = \int_{-\infty}^{\infty} f(x, z - x) dx, \quad x \in \mathbb{R}.$$

3. [16 marks] X, Y are independent random variables having $N(0, 1)$ distribution. Find the joint probability density function of $|X + Y|, |X - Y|$.

4. [6+6+6 marks] X_1, X_2, X_3 are independent standard normal random variables. Indicating clearly the results you are using, find the distributions of

(i) $X_1^2 + X_2^2 + X_3^2$, (ii) $-X_1/X_2$, (iii) $(X_1^2 + X_2^2)/2X_3^2$.

5. [6+9 marks] (i) Find the characteristic function of the binomial distribution with parameters n and p.

(ii) For $n = 1, 2, \ldots$ let X_n have a binomial distribution with parameters n and p_n. Suppose $\lim_{n\to\infty} np_n = \lambda > 0$. Let X have a Poisson distribution with parameter λ. Let $F_n, n = 1, 2, \ldots$ and F denote the respective distribution functions. Using characteristic functions show that $F_n(x) \to F(x)$ at every continuity point x of F.

6. [13+12 marks] For $a \geq 0, n = 1, 2, \ldots$ put

$$g(n,a) = \sum_{k \leq na} e^{-n} \frac{n^k}{k!}.$$

(i) Using Chebyshev's inequality show that

$$\begin{aligned} \lim_{n\to\infty} g(n,a) &= 0, \quad \text{if} \quad a < 1, \\ &= 1, \quad \text{if} \quad a > 1. \end{aligned}$$

(ii) Using the central limit theorem show that

$$\lim_{n\to\infty} g(n,1) = \frac{1}{2}.$$